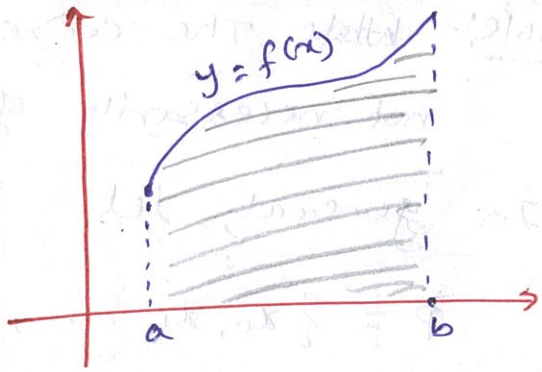
RIEMANN INTEGRATION

The "definite integral" is the key tool in Calculus for defining and calculating quantities such as areas, volumes, lengths of curved paths, probabilities, and the weights of various objects.

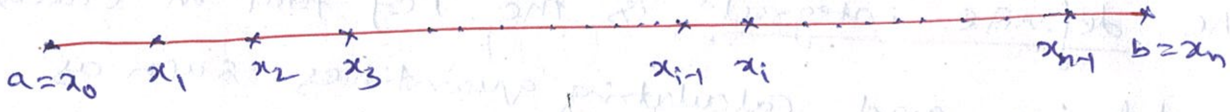
Historically, the idea of integration first arose in the work of "Archimedes (287–212 BC)" on the "quadrature of a parabola". This can be extended to the general problem of determining the "area" of a wide class of planar regions such as the region $R = \{(x,y) : a \le x \le b$ and $0 \le y \le f(x)\}$ under a curve $y = f(x)$, where $f : [a,b] \longrightarrow \mathbb{R}$ is a bounded function such that $f \geq 0$.



A formal set-up for these is the theory of "Riemann Integration".

We first need some basic definitions and terminology. Let $a, b \in \mathbb{R}$ with $a < b$.

A Partition of the interval $[a,b]$ is a set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of finitely many points of $[a,b]$ $\ni$ $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.



Examples:-

1. $P = \{a, b\}$ is a Partition of $[a,b]$.

2. $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ is a Partition of $[0,1]$.

3. For each $n \in \mathbb{N}$, the set

$$P_n = \left\{a,\ a+\frac{(b-a)}{n},\ a+\frac{2(b-a)}{n},\ ---,\ a+\frac{(n-1)(b-a)}{n}, b\right\}$$

is a partition of $[a,b]$ into n equal Parts.

This is sometimes called the Equidistant Partition of $[a,b]$ in n parts.

Note:- ~~While~~ The consecutive points of a Partition are not necessarily of equidistant.

In general, let

$$P = \{x_0, x_1, ---, x_n\}$$ be any partition of $[a,b]$,

where $$a = x_0 < x_1 < --- < x_n = b.$$

We call $[x_{i-1}, x_i]$ the i^{th} subinterval of the partition P. Its length $= x_i - x_{i-1}$ is denoted by δ_i or Δ_i. $\|P\| = \max\{\delta_1, \delta_2, ---, \delta_n\}$.

Given any bounded function $f : [a,b] \to \mathbb{R}$, let us define

$$m_i(f) := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

and

$$M_i(f) := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}, \quad i = 1, 2, \ldots, n.$$

Since f is bounded function on $[a,b]$, we define

$$m(f) := \inf\{f(x) \mid x \in [a,b]\}$$

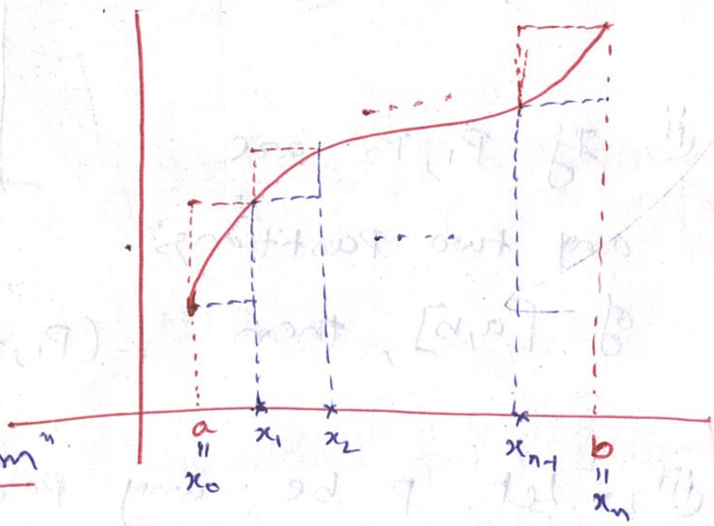
and

$$M(f) := \sup\{f(x) \mid x \in [a,b]\}$$

Note that

$$m(f) \le m_i(f) \le M_i(f) \le M(f) \quad \text{for each } i = 1, 2, \ldots, n.$$

Next, we define quantities that correspond to the sums of areas of the "lower rectangles" and the "upper rectangles".



Given a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a,b]$ and a bounded function $f : [a,b] \to \mathbb{R}$, we define the "lower sum" of f w.r.t P to be

$$L(P,f) = \sum_{i=1}^{n} m_i(f) \cdot (x_i - x_{i-1}).$$

and the "<u>upper sum</u>" of f w.r.t P to be

$$U(P,f) := \sum_{i=1}^{n} M_i(f)\,(x_i - x_{i-1}).$$

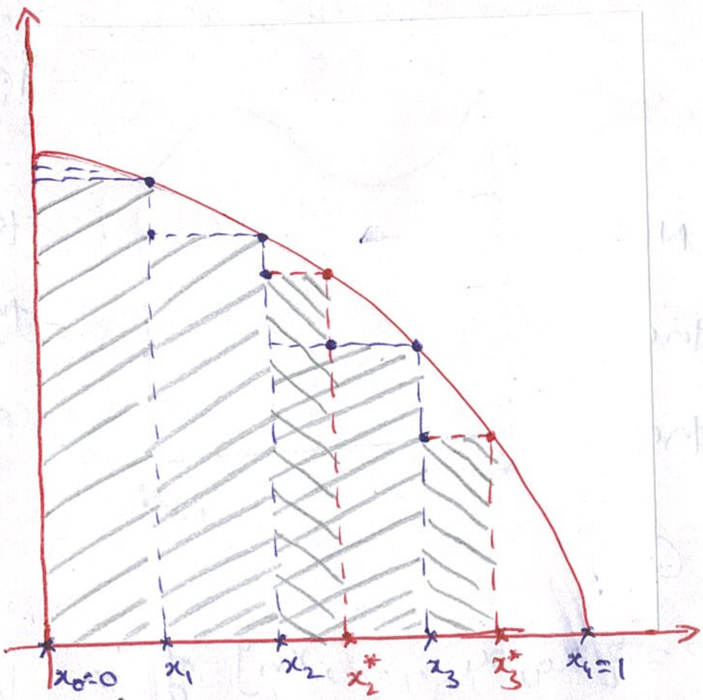
<u>Results:-</u>

Let $f:[a,b] \rightarrow \mathbb{R}$ be a bounded function.

i) If P is a partition of $[a,b]$ and P^* is a refinement of P, i.e., $P \subseteq P^*$, then

$$L(P,f) \leq L(P^*,f) \quad \& \quad U(P,f) \geq U(P^*,f).$$

In short, lower sums increase and the upper sums decrease as the partition becomes finer & finer.



ii) If P_1, P_2 are any two partitions of $[a,b]$, then $L(P_1,f) \leq U(P_2,f)$.

iii) Let P be any partition of $[a,b]$. Then

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a).$$

<u>Proof's are Easy!</u>

Def:- Let $f:[a,b] \longrightarrow \mathbb{R}$ be a bounded function and $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a,b]$.

The lower Riemann Integral of f on $[a,b]$ is defined as $\sup\{L(P,f) \mid P \text{ is a partition of } [a,b]\}$, and is denoted by $\underline{\int_a^b} f(x)\,dx$.

ie, $$\underline{\int_a^b} f(x)\,dx = \sup\{L(P,f) \mid P \text{ a partition of } [a,b]\}$$
$$= \sup_P L(P,f)$$

The upper Riemann Integral of f on $[a,b]$ is defined as $\inf\{U(P,f) \mid P \text{ a partition of } [a,b]\}$, and is denoted by $\overline{\int_a^b} f(x)\,dx$.

ie, $$\overline{\int_a^b} f(x)\,dx = \inf\{U(P,f) \mid P \text{ a partition of } [a,b]\}$$
$$= \inf_P U(P,f).$$

Def:- (Riemann Integrable) A bounded fn. $f:[a,b] \to \mathbb{R}$ is said to be "Riemann Integrable" over $[a,b]$ if
$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx,$$
and it is denoted by $\int_a^b f(x)\,dx$.

Remarks:

1. If $f : [a,b] \to \mathbb{R}$ is a bounded function, then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx.$$

2. A bounded function f is Riemann integrable on $[a,b]$ $\iff$ $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

3. If a bounded function f is such that

$$\underline{\int_a^b} f(x)\,dx \ne \overline{\int_a^b} f(x)\,dx,$$ then f is not Riemann integrable on $[a,b]$.

Examples:

1. A constant function is Riemann integrable on $[a,b]$.

Sol:- Let $f(x) = K \ \forall\ x \in [a,b]$, where $K \in \mathbb{R}$ be a constant.

Clearly f is bounded on $[a,b]$ and $\inf f = \sup f = K$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition on $[a,b]$.

Let m_r, M_r be the inf and sup of f on I_r, where $I_r = [x_{r-1}, x_r]$.

Since $f(x) = K \ \forall x \in [a,b]$, $m_r = M_r = K$.

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r = K \sum_{r=1}^{n} \delta_r = K(b-a)$$

$$\& \quad U(P,f) = \sum_{r=1}^{n} M_r \delta_r = K \sum_{r=1}^{n} \delta_r = K(b-a).$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \sup_P \{L(P,f)\} = K(b-a).$$

Similarly, $$\overline{\int_a^b} f(x)\,dx = \inf_P \{U(P,f)\} = K(b-a).$$

$$\therefore \int_a^b f(x)\,dx = \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = K(b-a).$$

$\therefore$ f is Riemann integrable on $[a,b]$. &

$$\int_a^b K\,dx = K(b-a).$$

2. The function $f(x) = \begin{cases} 1 & \text{when } x \in \mathbb{Q} \\ -1 & \text{when } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$, is not Riemann integrable on $[a,b]$.

Sol:- By the def. of the function, f,

$$-1 \le f(x) \le 1 \quad \forall x \in [a,b].$$

$\therefore$ f is bounded on $[a,b]$ and $\inf f = -1$, $\sup f = 1$.

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a Partition of $[a,b]$ and m_r, M_r be the inf and sup of f on $I_r = [x_{r-1}, x_r]$.

$\therefore m_r = -1$ and $M_r = 1$ for $r = 1, 2, \ldots, n$.

$$\therefore L(P,f) = \sum_{r=1}^{n} m_r \delta_r = -1 . \sum_{r=1}^{n} \delta_r = (-1)(b-a).$$

and $$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 . \delta_r = (b-a).$$

$\because L(P,f) = -(b-a)$ is constant, $\underline{\int_a^b} f(x)\,dx = -(b-a)$

Since $U(P,f) = (b-a)$ is constant, $\overline{\int_a^b} f(x)\,dx = (b-a)$.

$\therefore$ Lower and upper integrals exist but are not equal.

$\Rightarrow$ f is not Riemann integrable on $[a,b]$.

Note:- 1. A function need not be Riemann integrable on $[a,b]$, even though it is bounded on $[a,b]$.

2. If f is Riemann integrable on $[a,b]$ and m, M are the infimum and supremum of f on $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

3. If $f:[a,b] \to \mathbb{R}$ is a bounded function, then for each $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$

(i) $U(P,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon$ and

(ii) $L(P,f) > \underline{\int_a^b} f(x)\,dx - \varepsilon$ for each

$P \in \mathcal{P}[a,b]$ with $\|P\| < \delta$.

This result is known as Darboux's theorem.

$\mathcal{P}[a,b]$ ↓ Set of all Partitions on $[a,b]$

4. Let $f:[a,b] \to \mathbb{R}$ be a bounded function. As the norm of a partition, $\|P\|$, becomes small, the number of partition points becomes large in such a way that $n \to \infty$ as $\|P\| \to 0$.

Hence
$$\overline{\int_a^b} f(x)\,dx = \lim_{\|P\| \to 0} U(P,f) = \lim_{n \to \infty} U(P,f)$$

Similarly,
$$\underline{\int_a^b} f(x)\,dx = \lim_{\|P\| \to 0} L(P,f) = \lim_{n \to \infty} L(P,f).$$

* very useful in working out Problems on integrability of functions.

A Necessary and Sufficient Condition for Integrability

A bounded function $f:[a,b]\to\mathbb{R}$ is Riemann integrable on $[a,b]$ iff for each $\varepsilon>0$ $\exists$ a partition of P of $[a,b]$ such that

$$0 \le U(P,f) - L(P,f) < \varepsilon .$$

Proof:-

Necessary Part:

Let f be Riemann integrable on $[a,b]$.

$$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx \quad \text{--- (i)}$$

Let $\varepsilon>0$.

By Darboux's theorem $\exists\ \delta>0 \ni$

$$U(P,f) < \overline{\int_a^b} f(x)\,dx + \varepsilon/2 \quad \text{--- (ii)}$$

and

$$L(P,f) > \underline{\int_a^b} f(x)\,dx - \varepsilon/2 \quad \text{--- (iii)}$$

for each $P \in \mathcal{P}[a,b]$ with $\|P\|<\delta$.

From (i) & (ii),

$$U(P,f) < \int_a^b f(x)\,dx + \varepsilon/2 .$$

From (i) & (iii), $\displaystyle\int_a^b f(x)\,dx < L(P,f) + \varepsilon/2 .$

$\therefore\ U(P,f) < \left(L(P,f) + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2}$.

$\therefore\ U(P,f) - L(P,f) < \varepsilon$.

Also $U(P,f) - L(P,f) \geq 0$.

$\therefore\ 0 \leq U(P,f) - L(P,f) < \varepsilon$.

Sufficient Part:

Let for each $\varepsilon > 0$ $\exists\ P \in \mathcal{P}[a,b]$

such that

$$0 \leq U(P,f) - L(P,f) < \varepsilon.$$

By def. $\overline{\int_a^b} f(x)dx = \inf\{U(P,f) \mid P \in \mathcal{P}[a,b]\}$

$$\Rightarrow \overline{\int_a^b} f(x)dx \leq U(P,f).$$

By def. $\underline{\int_a^b} f(x)dx = \sup\{L(P,f) \mid P \in \mathcal{P}[a,b]\}$

$$\Rightarrow \underline{\int_a^b} f(x)dx \geq L(P,f).$$

$$\Rightarrow -\underline{\int_a^b} f(x)dx \leq -L(P,f).$$

$$\therefore\ 0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx \leq U(P,f) - L(P,f) < \varepsilon.$$

$\therefore$ For each $\varepsilon > 0$, we have $0 \leq \overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \varepsilon$.

$$\Rightarrow \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx.$$

$\therefore$ f is Riemann integrable on $[a,b]$.

Examples:-

1. Prove that $f(x) = x^2$ is integrable on $[0, a]$ and $\int_0^a x^2\,dx = \frac{a^3}{3}$.

Sol:- $f(x) = x^2$ is bounded on $[0, a]$.

Consider the partition $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \dots, \frac{ra}{n}, \dots, \frac{na}{n}\right\}$

$I_r = r^{th}$ subinterval $= \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$.

Length of each subinterval $= \delta_r = \frac{a}{n}$.

Since $f(x) = x^2$ is increasing function in $[0, a]$,

$$M_r = \sup \text{ of } f \text{ in } I_r = \left(\frac{ra}{n}\right)^2 = \frac{r^2a^2}{n^2}.$$

$$m_r = \inf \text{ of } f \text{ in } I_r = \left(\frac{(r-1)a}{n}\right)^2 = \frac{(r-1)^2a^2}{n^2}.$$

$$\therefore U(P,f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r^2a^2}{n^2} \times \frac{a}{n} = \frac{a^3}{n^3}\cdot\sum_{r=1}^{n} r^2$$

$$= \frac{a^3}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}.$$

$$L(P,f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{(r-1)^2a^2}{n^2} \times \frac{a}{n}.$$

$$= \frac{a^3}{n^3}\sum_{r=1}^{n}(r-1)^2 = \frac{a^3}{n^3}\cdot\frac{(n-1)n(2n-1)}{6}$$

$$\therefore \int_0^a f(x)\,dx = \lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} \frac{a^3}{6}\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) = \frac{a^3}{3}.$$

$$\& \ \overline{\int_0^a} f(x)\,dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty} \frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) = \frac{a^3}{3}$$

$$\therefore \int_{\underline{0}}^{a} f(x)\,dx = \int_{0}^{\bar{a}} f(x)\,dx = \frac{a^3}{3}.$$

$\therefore f(x) = x^2$ is integrable on $[0,a]$ & $\int_0^a x^2 dx = \frac{a^3}{3}$.

Properties :-

1. If $f:[a,b] \to \mathbb{R}$ is continuous on $[a,b]$, then f is integrable on $[a,b]$.

Proof :- f is continuous on $[a,b] \Rightarrow f$ is bounded on $[a,b]$.

f is continuous on $[a,b] \Rightarrow$ for each $\varepsilon > 0$ $\exists$ a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$ $\ni$

$$|f(y_r) - f(z_r)| < \frac{\varepsilon}{b-a} \quad \text{for } y_r, z_r \in I_r,\ r = 1,2,\dots,n.$$

For the partition P, let m_r, M_r be the infimum and supremum of f in I_r.

f is continuous on $I_r \Rightarrow \exists\ \alpha_r, \beta_r \in I_r \ni$

$m_r = f(\alpha_r)$, $M_r = f(\beta_r)$ [$\because$ a cont fn. on a closed & bounded interval attains its bounds.]

$$\therefore M_r - m_r = |f(\alpha_r) - f(\beta_r)|$$
$$< \frac{\varepsilon}{b-a} \quad \text{for } r = 1,2,\dots,n.$$

$$\therefore U(P,f) - L(P,f) = \sum_{r=1}^{n} (M_r - m_r)\delta_r < \frac{\varepsilon}{b-a}\cdot(b-a) = \varepsilon.$$

$\therefore f$ is integrable on $[a,b]$

Note:- (i) There exist functions which are integrable but not continuous.

Ex:- $f(x) = \begin{cases} -1 & \text{when } -1 \le x \le 0 \\ 1 & \text{when } 0 < x \le 1 \end{cases}$

f is bounded on $[-1,1]$ and it is discontinuous only at $x=0$.

This fn. is Riemann integrable on $[-1,1]$.

2. If $f:[a,b] \to \mathbb{R}$ is monotonic on $[a,b]$, then f is integrable on $[a,b]$.

3. If the set of points of discontinuity of a bounded function $f:[a,b] \to \mathbb{R}$ is finite, then ~~f~~ f is Riemann integrable on $[a,b]$.

4. If the set of points of discontinuity of a bounded function $f:[a,b] \to \mathbb{R}$ has a finite number of limit points, then f is integrable on $[a,b]$.

Note:- A function may have infinitely many discontinuous points, but if the set of all discontinuous pts have a finite number of limit points, then f is integrable on $[a,b]$.

Examples:-

1. Prove that $f(x) = [x]$ is integrable on $[0,3]$.

Sol:- $[x]$ = the greatest integer not greater than x.

Since $x \in [0,3]$, $f(x) = [x]$ is bounded on $[0,3]$

Further, $f(x) = [x]$ is discontinuous at $x = 1, 2$.

$\therefore$ $f(x) = [x]$ is bounded and has finite number of points of discontinuity.

$\therefore$ f is integrable on $[0,3]$.

2. $$f(x) = \begin{cases} 1/2^n & \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}; \ n = 0, 1, 2, \ldots \\ 0 & \text{when } x = 0. \end{cases}$$

Prove that f is integrable on $[0,1]$.

Sol:- For $n = 0$, $f(x) = 1$, when $\frac{1}{2} < x \le 1$

$n = 1$, $f(x) = \frac{1}{2}$ when $\frac{1}{2^2} < x \le 1/2$

$n = 2$, $f(x) = \frac{1}{2^2}$ when $\frac{1}{2^3} < x \le 1/2^2$

$\vdots$

$n = k$, $f(x) = \frac{1}{2^k}$ when $\frac{1}{2^{k+1}} < x \le \frac{1}{2^k}$

$\vdots$

and $f(0) = 0$.

$\therefore$ For all $x \in [0,1]$, $f(x) \in [0,1]$ hence f is bounded on $[0,1]$.

Also, f has $\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots, \frac{1}{2^n}, \ldots\}$ infinite set of points of discontinuity with unique limit point 0.

$\therefore$ f is integrable on $[0,1]$.

Some more properties of integrable functions:

1. If f is Riemann integrable on $[a,b]$ then

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

2. If f is Riemann integrable on $[a,b]$, then $-f$ also Riemann integrable on $[a,b]$, and

$$\int_a^b (-f)(x)\,dx = -\int_a^b f(x)\,dx.$$

Note:- If f is Riemann integrable on $[a,b]$, we can write as $f \in \mathcal{R}[a,b]$.

3. If $f \in \mathcal{R}[a,b]$ and $k \in \mathbb{R}$, then $kf \in \mathcal{R}[a,b]$ and

$$\int_a^b (kf)(x)\,dx = k\int_a^b f(x)\,dx.$$

4. If $f \in \mathcal{R}[a,b]$, then $|f| \in \mathcal{R}[a,b]$.

Note:- The converse is not true. i.e, if $|f| \in \mathcal{R}[a,b]$, then f need not be integrable on $[a,b]$.

Ex:- $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ on $[0,1]$.

$|f| = 1 \ \forall x \in [0,1]$ is integrable on $[0,1]$, but f is not integrable.

5. If $f, g \in \mathcal{R}[a,b]$, then $f+g \in \mathcal{R}[a,b]$ and

$$\int^b (f+g)(x)\,dx = \int^b f(x)\,dx + \int^b g(x)\,dx.$$

6. If $f \in \mathcal{R}[a,b]$, then $f^2 \in \mathcal{R}[a,b]$.

Example:- Use the properties of integrable functions and show that if $f, g \in \mathcal{R}[a,b]$, then $f \cdot g \in \mathcal{R}[a,b]$.

Sol:- Hint:-

$$f \cdot g = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$$

Note:- Even though f, g are not integrable on $[a,b]$, $f \cdot g$ may be integrable on $[a,b]$.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ -1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad \text{on } [0,1]$$

& $$g(x) = \begin{cases} -1 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \quad \text{on } [0,1],$$

then f, g are not integrable but $f \cdot g$ is integrable.

7. If $f, g \in \mathcal{R}[a,b]$ and $\exists\, t > 0 \ni |g(t)| \geq t$ $\forall\, t \in [a,b]$, then $\frac{f}{g} \in \mathcal{R}[a,b]$.

8. If $f \in \mathcal{R}[a,c]$, $f \in \mathcal{R}[c,b]$ and $a < c < b$, then $f \in \mathcal{R}[a,b]$.

9. If $f \in \mathcal{R}[a,b]$ and $a < c < b$, then $f \in \mathcal{R}[a,c]$, $f \in \mathcal{R}[c,b]$ and

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example:-

If f is continuous on $[a,b]$, $f(x) \geq 0 \ \forall x \in [a,b]$

and $\int_a^b f(x)\,dx = 0$, then $f(x) = 0 \ \forall x \in [a,b]$.

Sol:- If possible let $f(x) \neq 0 \ \forall x \in [a,b]$.

$\therefore \exists \ c \in [a,b] \ni f(c) \neq 0$ ie, $f(c) > 0$

(By hypothesis)

f is continuous at c

$\Rightarrow$ for $\varepsilon = \frac{f(c)}{2} > 0 \ \exists \ \delta > 0 \ni$

$$|f(x) - f(c)| < \frac{f(c)}{2} \text{ for } |x-c| < \delta$$

$$\Rightarrow \quad f(x) > \frac{f(c)}{2} \text{ for } |x-c| < \delta.$$

let $a < c < b$.

Then $\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx > \int_{c-\delta}^{c+\delta} \frac{f(c)}{2}\,dx = \frac{f(c)}{2} \cdot 2\delta > 0.$$

This is a contradiction to the fact that

$\int_a^b f(x)\,dx = 0$. (Even if $c = a$ or $c = b$, we can prove that $\int_a^b f(x)\,dx > 0$ and can arrive at the contradiction)

$\therefore f(x) = 0 \ \forall x \in [a,b]$.

Some inequalities:

1. If $f \in \mathcal{R}[a,b]$ and $f(x) \geq 0 \ \forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

2. If $f, g \in \mathcal{R}[a,b]$ and $f(x) \geq g(x) \ \forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

3. If $f \in \mathcal{R}[a,b]$ then $\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$

Proof: $f \in \mathcal{R}[a,b] \Rightarrow |f| \in \mathcal{R}[a,b]$

and $-|f| \in \mathcal{R}[a,b]$.

We have

$$-|f|(x) \leq f(x) \leq |f|(x) \quad \forall x \in [a,b]$$

$$\Rightarrow -|f(x)| \leq f(x) \leq |f(x)| \quad \forall x \in [a,b]$$

$$\therefore \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\therefore \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

4. If $f \in R[a,b]$ and $|f(x)| \leq K \ \forall x \in [a,b]$ where $K \in \mathbb{R}^+$, then

$$\left| \int_a^b f(x)\,dx \right| \leq K.(b-a).$$

Definition:

Let $f \in R[a,b]$. Then for each $t \in [a,b]$, $[a,t] \subseteq [a,b]$ and hence $f \in R[a,t]$.

$\therefore \int_a^t f(x)\,dx$ is well defined.

The function $\phi(t) = \int_a^t f(x)\,dx$, $t \in [a,b]$ is called the integral function of f.

ϕ is also called indefinite integral of f on $[a,b]$.

Note:- The integral function of f may also be defined as $\phi(t) = \int_t^b f(x)\,dx$, $t \in [a,b]$.

Results:-

1. If $f \in R[a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$, $t \in [a,b]$ is continuous on $[a,b]$.

Proof:- $f \in R[a,b] \Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow \exists\, K \in \mathbb{R}^+ \ni |f(x)| \leq K \ \forall x \in [a,b]$

Let $c \in [a,b]$ and $\varepsilon > 0$ be given.

Then $$|Q(c+h) - Q(c)| = \left| \int_a^{c+h} f(x)\,dx - \int_a^c f(x)\,dx \right|$$

$$= \left| \int_a^c f(x)\,dx + \int_c^{c+h} f(x)\,dx - \int_a^c f(x)\,dx \right|$$

$$= \left| \int_c^{c+h} f(x)\,dx \right| \le K\,|c+h-c| = K|h| < \varepsilon$$

$$\left(\text{if } |h| < \delta < \tfrac{\varepsilon}{K}\right)$$

$\therefore$ For $\varepsilon > 0$ $\exists$ $\delta \left(< \frac{\varepsilon}{K}\right) \ni |Q(c+h) - Q(c)| < \varepsilon$

for $|(c+h) - c| = |h| < \delta$.

$\therefore$ Q is <u>Cont</u> at $c \in [a,b]$.

$\therefore$ Q is <u>Cont</u> on $[a,b]$.

<u>Note</u>:- Even if f is discontinuous at points in $[a,b]$, the integral function Q is Continuous on $[a,b]$.

<u>Example</u>:- $f : [0,2] \to \mathbb{R}$ by $f(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 2 & \text{if } 1 \le x \le 2. \end{cases}$

clearly f is not continuous at $x = 1$.

If we define $Q(t) = \int_0^t f(x)\,dx = \begin{cases} t & \text{if } 0 \le t < 1 \\ 2t-1 & \text{if } 1 \le t \le 2. \end{cases}$

then $Q(t)$ is Continuous at $t = 1$ also.

2. If $f \in R[a,b]$ and f is Continuous at $c \in [a,b]$ then $Q(t) = \int_a^t f(x)\,dx$ $\forall\, t \in [a,b]$ is derivable at c and $Q'(c) = f(c)$.

"This result is sometimes referred as the first fundamental theorem of integral calculus."

Proof:- f is continuous at $c \in [a,b]$

$\Rightarrow$ for $\varepsilon > 0$ $\exists$ $\delta > 0$ $\ni$ $|f(x) - f(c)| < \varepsilon$

for $x \in [a,b]$ with $|x - c| < \delta$. --- (i)

Take h so that $|h| < \delta$.

Now $\quad \Phi(c+h) - \Phi(c) = \int_a^{c+h} f(x)\,dx - \int_a^{c} f(x)\,dx$

$$= \int_c^{c+h} f(x)\,dx \quad \text{--- (ii)}$$

and

$$\int_c^{c+h} f(c)\,dx = f(c) \int_c^{c+h} dx = f(c).h$$

$$\therefore f(c) = \frac{1}{h} \cdot \int_c^{c+h} f(c)\,dx \quad \text{--- (iii)}$$

$$\therefore \left| \frac{\Phi(c+h) - \Phi(c)}{h} - f(c) \right|$$

$$= \left| \frac{1}{h} \cdot \int_c^{c+h} f(x)\,dx - \frac{1}{h} \cdot \int_c^{c+h} f(c)\,dx \right| \quad \text{(by (ii) \& (iii))}$$

$$= \left| \frac{1}{h} \cdot \int_c^{c+h} (f(x) - f(c))\,dx \right|$$

$$\leq \frac{1}{|h|} \cdot \left| \int_c^{c+h} |f(x) - f(c)|\,dx \right|$$

$$< \frac{1}{|h|} \cdot \left| \int_c^{c+h} \varepsilon\,dx \right| \quad \text{(using (i))}$$

$$= \frac{1}{|h|} \cdot \varepsilon \cdot |h| = \varepsilon.$$

$\therefore$ Φ is derivable at c and $\Phi'(c) = f(c)$.

Def:- If $f \in R[a,b]$ and if $\exists\ \phi : [a,b] \to \mathbb{R}$ such that $\phi'(x) = f(x)\ \forall\ x \in [a,b]$, then ϕ is called a "primitive" or "antiderivative" of f.

Note:-

1. From first fundamental theorem of integral calculus, if f is continuous on $[a,b]$, then f possesses a primitive $\phi(t) = \int_a^t f(x)\,dx$ $\forall\ x \in [a,b]$.

2. Primitive of f is not unique. If ϕ is primitive of f, then $\phi + c$, $c \in \mathbb{R}$, is also a primitive of f.

3. The continuity of a function is not a necessary condition for the existence of a primitive.

Examples:-

1. Let $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = \sin x$.

   $f(x) = \sin x$ is cont. on $[a,b]$.

   $\therefore$ Primitive of $\sin x$ exists on $[a,b]$.

   If $\phi : [a,b] \to \mathbb{R}$ is defined by $\phi(x) = -\cos x$, then we know that $\phi'(x) = \sin x\ \forall\ x \in [a,b]$.

   $\therefore -\cos x$ is a primitive of $\sin x$ on $[a,b]$.

2. Consider a function ϕ defined on $[0,1]$ by

$$\phi(x) = \begin{cases} x^2 \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Then $$\phi'(x) = \begin{cases} 2x \sin(1/x) - \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We know that $\phi'(x)$ is not continuous at $x=0$.

If we take $f(x) = \phi'(x)$ in $[0,1]$, then $f(x)$ is not continuous in $[0,1]$ at $x=0$.

Eventhough $f(x)$ admits a Primitive $\phi(x)$ in $[0,1]$ it fails to be continuous in $[0,1]$.

2nd Fundamental Theorem of Integral Calculus:

If $f \in R[a,b]$ and ϕ is Primitive of f, then

$$\int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Proof: Since ϕ is Primitive of f on $[a,b]$,

$\phi'(x) = f(x) \quad \forall\, x \in [a,b]$. --- (i)

Since $f \in R[a,b]$ for $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$, and $x_{r-1} \le \xi_r \le x_r$, $r = 1, 2, \dots, n$,

we have $\lim\limits_{\|P\| \to 0} \sum\limits_{r=1}^{n} f(\xi_r) \cdot \delta_r = \int_a^b f(x)\,dx$ --- (ii)

ϕ is derivable on $[a,b]$

$\Rightarrow$ ϕ is continuous and derivable on $[x_{r-1}, x_r]$

$r = 1, 2, \ldots, n$

By Lagrange mean-value theorem:

$$\phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1}) \cdot \phi'(\xi_r),$$

for $\xi_r \in (x_{r-1}, x_r)$, $r = 1, 2, \ldots, n$.

$$\therefore \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} \phi'(\xi_r) \cdot \delta_r$$

$$= \sum_{r=1}^{n} f(\xi_r)\, \delta_r \quad \text{(using (i))}$$

$$\Rightarrow \phi(x_n) - \phi(x_0) = \sum_{r=1}^{n} f(\xi_r) \cdot \delta_r.$$

$$\therefore \lim_{\|P\| \to 0} [\phi(x_n) - \phi(x_0)] = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r) \cdot \delta_r.$$

$$\Rightarrow \phi(b) - \phi(a) = \int_a^b f(x)\, dx.$$

Note:- If ϕ' is continuous on $[a,b]$, then

$$\int_a^b \phi'(x)\, dx = \phi(b) - \phi(a).$$

Properties:-

1. If f is continuous on $[a,b]$, then $\exists$ $c \in (a,b)$ such that

$$\int_a^b f(x)\, dx = f(c)\,(b-a).$$

* This result is known as "Mean-value theorem".

Proof:- f is continuous on $[a,b]$

$\Rightarrow$ f is bounded on $[a,b]$ and $f \in R[a,b]$.

If m, M be the inf and sup of f on $[a,b]$, we know that $m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$.

$\therefore \exists\ \mu \in [m, M] \ni \int_a^b f(x)\,dx = \mu(b-a)$.

Since f is continuous on $[a,b]$ and $\mu \in [m, M]$

$\exists\ c \in (a,b) \ni f(c) = \mu$.

$$\therefore \int_a^b f(x)\,dx = f(c)(b-a).$$

Example:-

Prove that $\frac{\pi}{4} \le \int_0^{\pi/4} \sec x\,dx \le \frac{\pi}{2\sqrt{2}}$

Sol:- Consider $f(x) = \sec x$ on $[0, \pi/4]$.

clearly $\sec x$ is continuous on $[0, \pi/4]$ and hence integrable on $[0, \pi/4]$.

By above theorem, $\exists\ c \in [0, \pi/4] \ni$

$$\int_0^{\pi/4} \sec x\,dx = \cancel{\sec(\pi/4)}\ (\pi/4 - 0) \cdot \sec c.$$

$0 < c < \pi/4 \Rightarrow \cos 0 > \cos c > \cos(\pi/4)$

$\Rightarrow 1 < \sec c < \sqrt{2}$.

$\Rightarrow \frac{\pi}{4} < \frac{\pi}{4}\sec c < \frac{\pi\sqrt{2}}{4}$.

Also, $\left(\frac{\pi}{4}\right) \sec c = \frac{\pi}{4}$ for $c = 0$ and

$\left(\frac{\pi}{4}\right) \sec c = \frac{\pi\sqrt{2}}{4}$ for $c = \frac{\pi}{4}$.

Hence $\frac{\pi}{4} \le \int_0^{\pi/4} \sec x \, dx \le \frac{\pi}{2\sqrt{2}}$.

2. (~~First~~ Mean-Value Theorem)

If $f, g \in R[a,b]$ and g keeps the same sign on $[a,b]$, then $\exists\ \mu \in \mathbb{R}$ lying between the inf. and sup. of f $\ni$ $\int_a^b f(x) g(x)\, dx = \mu \int_a^b g(x)\, dx$

Proof:-

W.O.L.O.G, let $g(x) \geq 0 \ \forall x \in [a,b]$.

$f \in R[a,b] \Rightarrow f$ is bounded on $[a,b]$

$\Rightarrow m \le f(x) \le M \ \forall x \in [a,b]$,

where m, M are the inf & sup of f resply.

$\because g(x) \geq 0 \ \forall x \in [a,b]$,

$m. g(x) \le f(x). g(x) \le M. g(x)$

$\therefore \int_a^b m. g(x)\, dx \le \int_a^b f(x). g(x)\, dx \le \int_a^b M. g(x)\, dx$

$\Rightarrow m \int_a^b g(x)\, dx \le \int_a^b f(x). g(x)\, dx \le M. \int_a^b g(x)\, dx$.

$\therefore \exists\ \mu \in [m, M] \ni \int_a^b f(x) g(x)\, dx = \mu \int_a^b g(x)\, dx$.

Similarly, we can prove when $g(x) \le 0 \ \forall x \in [a,b]$.

Note:-

1. If f is continuous on $[a,b]$, $g \in R[a,b]$ and g keeps the same sign on $[a,b]$, then $\exists\ \xi_1 \in (a,b)$ $\ni$

$$\int_a^b f(x) \cdot g(x)\, dx = f(\xi_1) \int_a^b g(x)\, dx.$$

2. If we take $g(x) = 1\ \forall\, x \in [a,b]$, then g is integrable on $[a,b]$ and $g(x) \geq 0\ \forall\, x \in [a,b]$.

$\therefore$ By the mean-value theorem,

$$\int_a^b f(x) \cdot 1\, dx = \mu \cdot \int_a^b 1\, dx = \mu(b-a),$$

where $\mu \in [m, M]$, m, M being the inf and sup. of f on $[a,b]$, respectively.

Examples:-

1. Prove that $\frac{1}{\pi} \leq \int_0^1 \frac{\sin \pi x}{1+x^2}\, dx \leq \frac{2}{\pi}$.

Sol:- Take $f(x) = \frac{1}{1+x^2}$ and $g(x) = \sin \pi x$.

Clearly f, g are continuous on $[0,1]$ and hence integrable on $[0,1]$.

Also $g(x) = \sin \pi x$ is positive on $[0,1]$.

$\because$ f is decreasing on $[0,1]$,

$\inf f = f(1) = \frac{1}{2}$ and $\sup f = f(0) = 1$.

By first-mean value theorem, $\exists\ \mu \in [\frac{1}{2}, 1]$ such that

$$\int_0^1 \frac{\sin \pi x}{1+x^2}\, dx = \mu \int_0^1 \sin \pi x\, dx = f(\xi_1) \int_0^1 \sin \pi x\, dx$$

where $\xi_1 \in (0,1)$.

By the fundamental theorem,

$$\int_0^1 \sin\pi x \, dx = \frac{2}{\pi},$$

$$\therefore \int_0^1 \frac{\sin\pi x}{1+x^2} dx = f(\xi_1) \cdot \frac{2}{\pi}, \text{ where } 0 < \xi_1 < 1.$$

But $0 < \xi_1 < 1 \Rightarrow \frac{1}{2} \le f(\xi_1) \le 1$

$$\Rightarrow \frac{1}{2} \cdot \frac{2}{\pi} \le \frac{2}{\pi} \cdot f(\xi_1) \le 1 \cdot \frac{2}{\pi}$$

$$\therefore \frac{1}{\pi} \le \int_0^1 \frac{\sin\pi x}{1+x^2} dx \le \frac{2}{\pi}.$$

Change of Variable in an Integral:

If i, g is derivable and integrable on $[\alpha, \beta]$

ii, $g(\alpha) = a$, $g(\beta) = b$ where $a < b$,

iii, range of $g = [a,b]$ and

iv, f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = \int_\alpha^\beta f(g(t)) \cdot g'(t)\,dt.$$

Proof:- Firstly, let us prove that $\int_a^b f(x)\,dx$ and $\int_\alpha^\beta f(g(t)) \cdot g'(t)$ exists.

f is continuous on $[a,b] \Rightarrow f \in R[a,b]$.

g is derivable on $[\alpha,\beta] \Rightarrow g$ is continuous on $[\alpha,\beta]$

Since the range of g is $[a,b]$ & f is defined on $[a,b]$, $f \circ g$ is defined on $[\alpha,\beta]$.

$\because$ both g & f are continuous, $f \circ g$ is continuous & hence $f \circ g$ is $R[\alpha,\beta]$.

Now $f \circ g \in R[\alpha,\beta]$, $g' \in R[a,b]$

$$\Rightarrow \int_{\alpha}^{\beta} (f \circ g) \cdot g' \text{ exists.}$$

Since f is continuous on $[a,b] = g([\alpha,\beta])$, its primitive F exists $\ni$ $F'(x) = f(x)$ $\forall x \in [a,b]$

For $t \in [\alpha,\beta]$, if $G(t) = F(g(t))$, then by chain rule,

$$G'(t) = F'(g(t)) \cdot g'(t) = f(g(t)) \cdot g'(t)$$

$\therefore$ By fundamental theorem,

$$\int_{\alpha}^{\beta} f(g(t)) \cdot g'(t)\,dt = \int_{\alpha}^{\beta} G'(t)\,dt = G(\beta) - G(\alpha)$$

$$= F(g(\beta)) - F(g(\alpha)) = F(b) - F(a) = \int_a^b f(x)\,dx$$

*

Note:-

1. If f' exists in $[a,b]$, $f' \in R[a,b]$ and $f(x) \neq 0 \ \forall \ x \in [a,b]$, then

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \log \frac{f(b)}{f(a)}.$$

Outline proof /Hint:- $\because f \neq 0$, $\frac{1}{f}$ is cont on $[a,b]$ ($\because f'$ exists)

$\Rightarrow \frac{1}{f} \in R[a,b]$. $\therefore \frac{f'}{f} \in R[a,b]$

If $\phi(x) = \log|f(x)|$ on $[a,b]$, then $\phi'(x) = \frac{f'(x)}{f(x)}$ $\forall \ x \in [a,b]$. i.e., ϕ is a primitive of f'/f on $[a,b]$.

$\therefore$ By fundamental th,

$$\int_a^b \frac{f'(x)}{f(x)}\,dx = \phi(b) - \phi(a) = \log|f(b)| - \log|f(a)|.$$

Example:-

1. Using the substitution $x = \pi - t$, show that

$$\int_0^\pi x\,\phi(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi \phi(\sin x)\,dx$$

Sol:- Let $x = \pi - t \Rightarrow t = \pi$ when $x = 0$ and $t = 0$ when $x = \pi$.

Since $g(t) = \pi - t$ is defined on $[0,\pi]$,

$g([0,\pi]) = [\pi, 0]$ and $g'(t) = -1 \ \forall \ t \in [0,\pi]$

We know that $g \in R[0,\pi]$.

$\therefore$ By the theorem of change of variables,

$$\int_0^\pi x\, Q(\sin x)\,dx = \int_\pi^0 (\pi - t)\, Q(\sin(\pi - t))\,(-1)\,dt.$$

$$= \int_0^\pi (\pi - t)\, Q(\sin t)\,dt = \pi \int_0^\pi Q(\sin t)\,dt - \int_0^\pi t\, Q(\sin t)\,dt$$

$$= \pi \int_0^\pi Q(\sin x)\,dx - \int_0^\pi x\, Q(\sin x)\,dx$$

$$\therefore\ 2\int_0^\pi x\, Q(\sin x)\,dx = \pi \int_0^\pi Q(\sin x)\,dx$$

$$\Rightarrow \int_0^\pi x\, Q(\sin x)\,dx = \frac{\pi}{2} \int_0^\pi Q(\sin x)\,dx.$$